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Engineering entanglement of a general three-level system interacting with a correlated two-mode nonlinear coherent state

Mahmoud Abdel-Aty[1,a] and Abdel-Shafy F. Obada[2]

[1] Mathematics Department, Faculty of Science, South Valley University, 82524 Sohag, Egypt
[2] Mathematics Department, Faculty of Science, Al-Azhar University, Naser City, Cairo, Egypt

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Abstract. In this article a treatment of a three-level atom interacting with two modes of light in a cavity with arbitrary forms of nonlinearities of both the fields and the intensity-dependent atom-field coupling is presented. A factorization of the initial density operator is assumed, with the privileged field modes being in a pair-coherent state. We derive and illustrate an exact expression for the time evolution of the density operator, by means of which we identify and numerically demonstrate the region of parameters where significantly large entanglement can be obtained. We show that entanglement can be significantly influenced by different kinds of nonlinearities. The nonlinear medium yields the superstructure of atomic Rabi oscillation. We propose a generation of Bell-type states having a simple initial state preparation of the present system.

PACS. 42.65.Sf Dynamics of nonlinear optical systems; optimal instabilities, optical chaos and complexity, and optical spatio-temporal dynamics – 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.) – 03.67.Hk Quantum communication

1 Introduction

Quantum entanglement has been studied intensely in recent years due to its potential applications in quantum communication and information processing [1]. Sources offering a great variety of entangled states are required for the implementation of many quantum communication and computation protocols [2,3]. With quantum communication [4] in mind the choice of photon-states as qubits is especially appropriate, since they can be easily transfered over long distances. The standard source presently used in the lab is parametric down conversion in a crystal [5,6]. It is a reliable source of entangled twin-photons but the process is random and largely untailorable. Moreover, in practice its capability of generating entanglement is limited to states comprising only two photons.

Recently Gilchrist *et al.* [7] and Munro [8] showed how in a certain narrow regime, the pair-coherent state gives quantum mechanical predictions that are in disagreement with those of local hidden variable theories for a situation involving continuous quadrature phase amplitude measurements. This test could be achieved by binning the continuous position and momentum information into two categories and using the binary results in the strong Clauser Horne Bell inequality test [9]. The predicted violation was small (less than 2%). Such results were highly idealized, and assumed the preparation of a pair-coherent state as a start. On the other hand, there exist situations in which the transition between the upper and lower levels of an atom is mediated by two photons if the energy separation between the levels is close to twice the photon frequency. This process and its multiphoton counterparts are important because they can be used to study statistical properties of the optical field [10]. In this case a system with more than two levels should be considered because a third level is required to support the second resonance [11]. The theory of the single-mode two-photon system has been developed [12]. An extended treatment of two-mode three-level cavity quantum electrodynamics has been reported in the long review [13]. On the other hand there is growing interest in nonlinear quantum dynamics [14]. However, works dealing with the nonlinear quantum dynamics have been limited to the two-level atom [14–16]. It is therefore desirable to investigate the nonlinear interaction of the three-level atom with the excited field on the two-photon resonance transitions.

In a previous paper [17] it was supposed that the cavity is filled with a nonlinear medium and a single-mode cavity interacts with both the three-level atom and the

[a] e-mail: `abdelaty@uni-flensburg.de`
Present address: Institut für Mathematik, Universität Flensburg, Germany.

nonlinear medium. It is one purpose of the present essay to extend the previously cited treatment to study the problem of entanglement measure in terms of the quantum field entropy of the reduced single-particle density matrix for a system of a three-level bimodal field interacting with a pair-coherent state taking into account arbitrary forms of nonlinearities of both the field and the intensity-dependent atom-field coupling. In this two-photon regime, each atomic transition creates simultaneously one photon in each mode, so that one expects a strong quantum correlation between the modes. The material of this paper is arranged as follows: in Section 2, we discuss the construction and properties of the two-mode nonlinear coherent states. Section 3 includes a description of the quantum phase gate. In Section 4, we introduce the model and write the expression for the reduced density operator. In Section 5, we introduce the entanglement degree calculation when the nonlinearities effects are included. By a numerical computation, we examine the influence of the nonlinearities on the entanglement degree and the atomic level occupation probabilities for a pair-coherent state in Section 6. Finally, conclusions are presented in Section 7.

2 Nonlinear coherent state

The generalized annihilation (creation) operator associated with nonlinear coherent states are given by $\hat{A} = \hat{a}f(N)$; $\hat{A}^\dagger = f(\hat{N})\hat{a}^\dagger$, $\hat{N} = \hat{a}^\dagger\hat{a}$ where $f(\hat{N})$ is a reasonably well behaved real function and $\hat{a}^\dagger(\hat{a})$ is the harmonic oscillator creation (annihilation) operator. It can be easily verified that $\hat{A}^\dagger$, $\hat{A}$ and $\hat{N}$ satisfy the following nonlinear algebra: $[\hat{N},\hat{A}] = -\hat{A}$, $[\hat{N},\hat{A}^\dagger] = \hat{A}^\dagger$, and $[\hat{A},\hat{A}^\dagger] = (\hat{N}+1)f^2(\hat{N}+1) - \hat{N}f^2(\hat{N})$. Clearly the nature of the nonlinear algebra depends on the choice of the nonlinearity function $f(\hat{N})$. Nonlinear coherent states $|\alpha\rangle$ are then defined as right eigenstates of the generalized annihilation operator $\hat{A}$,

$$\hat{A}|\alpha\rangle = \alpha|\alpha\rangle, \tag{1}$$

where α is an arbitrary complex number. In analogy to the definition of the one-mode nonlinear coherent states, the two-mode nonlinear coherent state is defined as

$$\hat{a}\hat{b}f(\hat{N}_a,\hat{N}_b)|\alpha,f,q\rangle = \alpha|\alpha,f,q\rangle, \tag{2}$$

where $\hat{a}$ and $\hat{b}$ are boson annihilation operators; $f(\hat{N}_a,\hat{N}_b)$ is the function of the number operator $\hat{N}_a = \hat{a}^\dagger\hat{a}$ and $\hat{N}_b = \hat{b}^\dagger\hat{b}$; q is the photon number difference between two modes of the field. The pair coherent state $|\zeta,q\rangle$, is an important correlated two-mode nonlinear coherent state, if the initial state of the two-mode optical field is prepared in the pair-coherent state defined as the eigenstate of pair annihilation operators $\hat{a}_1\hat{a}_2$ for two modes [18],

$$\hat{a}_1\hat{a}_2|\zeta,q\rangle = \zeta|\zeta,q\rangle, \tag{3}$$

where ζ is a complex number and q is the degeneracy parameter, such that

$$\left(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2\right)|\zeta,q\rangle = q|\zeta,q\rangle, \tag{4}$$

which implies that, whenever photons are either created in pairs or destroyed in pairs, the difference in the number of photons remains constant. The parameter q will be zero when pair creation starts from vacuum, *i.e.* the pair coherent states are only a special case of the abovementioned two-mode nonlinear coherent states. Without loss of generality, q can be assumed to be positive, where q, which remains constant is the difference of the number of two-mode photons. The pair coherent state takes the following form,

$$|\zeta,q\rangle = N_q\sum_{n=0}^{\infty}\frac{\zeta^n}{\sqrt{n!\,(n+q)!}}|n+q,n\rangle, \tag{5}$$

where $|n,m\rangle$ is such that $\hat{a}_1^\dagger\hat{a}_1|n,m\rangle = n|n,m\rangle$ and $\hat{a}_2^\dagger\hat{a}_2|n,m\rangle = m|n,m\rangle$, and the normalization constant N_q is determined by the condition $\langle\zeta,q|\zeta,q\rangle = 1$. We obtain

$$\begin{aligned}N_q &= 1/\sqrt{\sum_{n=0}^{\infty}\frac{|\zeta|^{2n}}{n!\,(n+q)!}}\\ &= \left\{(\mathrm{i}|\zeta|)^{-q}J_q(2\mathrm{i}|\zeta|)\right\}^{-\frac{1}{2}},\end{aligned} \tag{6}$$

where $J_q(x)$ is Bessel's function. The probability of finding n photons in mode 2 and $n+q$ photons in mode 1 is

$$P_n = |\langle n,n+q|\zeta,q\rangle|^2 = N_q^2\frac{|\zeta|^{2n}}{n!\,(n+q)!}, \tag{7}$$

which is sub-Poissonian. In addition to the sub-Poissonian statistics, the pair coherent states also possess other non classical features, such as the correlation in the number fluctuations, squeezing, and violations of Cauchy-Schwartz inequalities. Many other features of the pair coherent state, including the possibility of its generation, are investigated in reference [18].

3 Phase gate

For the convenience of the reader, we first briefly recapitulate the relevant known facts about the phase-gate and its relation to the quantum computations. The difficulty of building a quantum computer was greatly diminished when it was realized that a network of quantum phase gates operating in the product space of two qubits, single bit rotations, and single bit phase shift gates can constitute a universal quantum computer [19,20]. The quantum phase gate simply gives the product state of two qubits a phase shift depending on the values of each qubit. In other words, the quantum phase gate performs the operation $|00\rangle$, $\mathrm{e}^{\mathrm{i}\alpha}|01\rangle$, $\mathrm{e}^{\mathrm{i}\beta}|10\rangle$, $\mathrm{e}^{\mathrm{i}\gamma}|11\rangle$, in the computational basis of the two qubits. Provided that $\alpha + \beta \neq \gamma$ (mod 2π) a network of quantum phase gates supplemented with single bit gates can mimic the operation of any other unitary operator acting on the qubits. Recently an implementation of a quantum phase gate has been demonstrated [21] utilizing Rydberg states and a photon in a microwave cavity. Here one can explain how a quantum phase gate that operates in the product space of the polarizations of two photons can be constructed by using the optical Kerr effect. The photons are made to interact as they pass through a material with a third-order nonlinear susceptibility. These Kerr materials are used for a wide variety of optical applications. In the presence of a superposition of electromagnetic waves at different frequencies and/or in different directions, these materials are used in four-wave mixing applications such as frequency conversion, phase conjugation, real time holography, and image correlation. In the presence of a wave at a single frequency, the refractive index of such materials is intensity dependent and gives rise to the phenomenon of self focusing [22]. When a superposition of waves is present, the optical Kerr effect produces an interaction in which the intensity of one frequency component influences the index of refraction of another frequency component. As described by Mandel and Wolf in [23], this effect can be used to perform quantum non-demolition and back-action evading measurements, during which the intensity of one frequency component can be used to control the phase of another without altering the photon number of either component. Thus without loss of photon number, the frequency components can become entangled in a way that lends itself well to quantum computations.

4 The model

We begin with the description of the system. We consider here a three-level atom in $\Lambda-$configuration with unequally spaced levels, coupled to a quantized multimode electromagnetic field in the rotating wave approximation in an ideal cavity ($Q = \infty$). The transition frequencies from the upper state $|a\rangle$ to the lower states $|b\rangle$ and $|c\rangle$ are ω_{ab} and ω_{ac}, respectively. The transitions are connected by electric-dipole moments μ_{ab} and μ_{ac}, whereas the transition $|b\rangle \longleftrightarrow |c\rangle$ is forbidden in the electric-dipole approximation. Also, possible forms of nonlinearities for both the two-mode field and the intensity-dependent atom-fields couplings in a perfect cavity are included. Furthermore, we assume that the cavity modes interact with both the atom and the nonlinear medium. However, a real cavity cannot be ideal. But in reference [24] the influence of a cavity with finite bandwidth at nonzero temperature was studied and it was shown that for new available experimental values of $Q = 2 \times 10^{10}$ and the temperature $T = 0.5$ K the effect of the bandwidth and the temperature are negligible until the time $t \sim 10^{-3}$ ($\lambda t = 30$) from the start of the interaction. We assume for simplicity the two-photon resonance condition, but individual modes are allowed to be detuned from the upper atomic level by an arbitrary amount Δ. With these assumptions, the total interaction Hamiltonian in the rotating wave approximation can be written as (we adopt $\hbar = 1$)

$$\begin{aligned}\hat{H}_{\mathrm{in}} = \Delta|a\rangle\langle a| + \Re(\hat{a}_1^\dagger\hat{a}_1, \hat{a}_2^\dagger\hat{a}_2) \\ +\lambda_1(\hat{a}_1^\dagger f_1(\hat{a}_1^\dagger\hat{a}_1)|b\rangle\langle a| + f_1(\hat{a}_1^\dagger\hat{a}_1)\hat{a}_1|a\rangle\langle b|) \\ +\lambda_2(\hat{a}_2^\dagger f_2(\hat{a}_2^\dagger\hat{a}_2)|c\rangle\langle a| + f_2(\hat{a}_2^\dagger\hat{a}_2)\hat{a}_2|a\rangle\langle c|).\end{aligned} \quad (8)$$

We denote by $\hat{a}_i$ and $\hat{a}_i^\dagger$, the annihilation and the creation operators for the ith mode of the cavity field, respectively. The detuning $\Delta = \omega_{ab} - \Omega_1 = \omega_{ac} - \Omega_2$, ($\omega_{ij} = \omega_i - \omega_j$), and the operators $|i\rangle\langle j|$, ($i, j = a, b, c$) are the lowering and rising operators between level i and j. Also, λ_1 and λ_2 are the usual coupling constant corresponding to the coupling constants between the field and the atom. $f_i(\hat{n}_i)$ and $\Re(\hat{n}_1, \hat{n}_2)$ are Hermitian operators functions of photon number operators of the two modes, such that $\lambda_i f(n_i)$ represents an arbitrary intensity-dependent atom-field coupling, while $\Re(\hat{n}_1, \hat{n}_2)$ denotes the two-mode fields nonlinearity which can model Kerr-like medium nonlinearity as will be discussed later. A similar technique has been demonstrated by Sinatra *et al.* in [25], in their experiment the two fields are coupled through a gas of ^{87}Rb where the photons interact with a $\Lambda-$type three-level system. A similar model has been considered earlier but for $\Re(\hat{a}_1^\dagger\hat{a}_1, \hat{a}_2^\dagger\hat{a}_2) = 0$ [26]. If we assume that at time $t = 0$ the bimodal field-atom system is in a pure state, then the initial density operator of system can be given by $\rho(0) = \rho_f(0) \otimes \rho_a(0)$, where the initial bimodal field is given by

$$\rho_f(0) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} b_{n_1,n_2} b_{m_1,m_2} |n_1, n_2\rangle\langle m_1, m_2|, \quad (9)$$

with $b_{n_1,n_2} = b_{n_1}b_{n_2}$, b_{n_i} describes the amplitude of state $|n_i\rangle$ of the ith mode. The initial atomic density operator is assumed as

$$\rho_a(0) = |a\rangle\langle a|. \quad (10)$$

At any time $t > 0$, the time evolution of the statistical operator $\hat{\rho}(t)$ is given by

$$\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[\hat{H}(t), \hat{\rho}(t)]. \quad (11)$$

For the initial condition equation (9), we identify a general solution to equation (11), which is valid for any class of nonlinearities of both the intensity-dependent atom-field coupling and nonlinear medium. The time-dependent analytical solution for the density matrix $\hat{\rho}(t)$ is given by

$$\begin{aligned}\hat{\rho}(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big[\Im_1(n)\Im_1^*(m)|\Psi_1(n)\rangle\langle\Psi_1(m)| \\ +\Im_1(n)\Im_2^*(m)|\Psi_1(n)\rangle\langle\Psi_2(m)| + \Im_1(n)\Im_3^*(m) \\ \times|\Psi_1(n)\rangle\langle\Psi_3(m)| + \Im_2(n)\Im_1^*(m)|\Psi_2(n)\rangle\langle\Psi_1(m)| \\ +\Im_2(n)\Im_2^*(m)|\Psi_2(n)\rangle\langle\Psi_2(m)| + \Im_2(n)\Im_3^*(m)| \\ \times\Psi_2(n)\rangle\langle\Psi_3(m)| + \Im_3(n)\Im_1^*(m)|\Psi_3(n)\rangle\langle\Psi_1(m)| \\ +\Im_3(n)\Im_2^*(m)|\Psi_3(n)\rangle\langle\Psi_2(m)| + \Im_3(n)\Im_3^*(m) \\ \times|\Psi_3(n)\rangle\langle\Psi_3(m)|\Big],\end{aligned} \quad (12)$$

where $\Im_i(n)$ and $|\Psi_i(n)\rangle$ are given by

$$\begin{pmatrix} \Im_1(n) \\ \Im_2(n) \\ \Im_3(n) \end{pmatrix} = \begin{pmatrix} A(n,t) \\ B(n,t) \\ C(n,t) \end{pmatrix}, \tag{13}$$

$$\begin{pmatrix} |\Psi_1(n)\rangle \\ |\Psi_2(n)\rangle \\ |\Psi_3(n)\rangle \end{pmatrix} = \begin{pmatrix} |n_1+1, n_2, b\rangle \\ |n_1, n_2, a\rangle \\ |n_1, n_2+1, c\rangle \end{pmatrix}, \tag{14}$$

with

$$\begin{pmatrix} A(n,t) \\ B(n,t) \\ C(n,t) \end{pmatrix} = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{pmatrix} \begin{pmatrix} \exp\{i\mu_1 t\} \\ \exp\{i\mu_2 t\} \\ \exp\{i\mu_3 t\} \end{pmatrix}. \tag{15}$$

In equations (12–14), n refers to n_1, n_2 and m refers to m_1, m_2. The quantities ξ_{ij}, in the above equations are given by

$$\begin{aligned} \xi_{1i} &= \xi_{3i} M_1^{-1} M_2^{-1} \left(\mu_i^2 - \mu_i[\Re_2 + \Re_3] + \Re_3\Re_2 - M_2^2\right), \\ \xi_{2i} &= -\xi_{3i} M_2^{-1} \left(\mu_i + \Re_3\right), \\ \xi_{3i} &= (\mu_{ik})^{-1}(\mu_{il})^{-1} M_2 b_{n_1,n_2} \left(\mu_k + \mu_l + \Re_2 + \Re_3\right), \end{aligned} \tag{16}$$

where $i \neq k \neq l, (\mu_{ij} = \mu_i - \mu_j)$. The coefficients μ_i $(i = 1,2,3)$ are given by the following expressions

$$\begin{aligned} \mu_1 &= -\frac{1}{3}\left[\gamma_1 + 2\left(\sqrt{\gamma_1^2 - 3\gamma_2}\right)\cos\varpi\right], \\ \mu_2 &= -\frac{1}{3}\left[\gamma_1 - (\cos\varpi + \sqrt{3}\sin\varpi)\left(\sqrt{\gamma_1^2 - 3\gamma_2}\right)\right], \\ \mu_3 &= -\frac{1}{3}\left[\gamma_1 - (\cos\varpi - \sqrt{3}\sin\varpi)\left(\sqrt{\gamma_1^2 - 3\gamma_2}\right)\right], \end{aligned} \tag{17}$$

where

$$\begin{aligned} \varpi &= \frac{1}{3}\cos^{-1}\left(\frac{9\gamma_1\gamma_2 - 2\gamma_1^3 - 27\gamma_3}{2\sqrt{(\gamma_1^2 - 3\gamma_2)^3}}\right), \\ \gamma_1 &= \Re_1 + \Re_2 + \Re_3, \\ \gamma_2 &= -\left[M_1^2 + M_2^2 - \Re_1\Re_2 - \Re_2\Re_3 - \Re_1\Re_3)\right], \\ \gamma_3 &= \Re_1\Re_2\Re_3 - M_1^2\Re_3 - M_2^2\Re_1, \\ M_i &= \lambda_i f_i(n_i)\sqrt{n_i + 1}, \\ \Re_1 &= \Re(n_1 + 1, n_2), \\ \Re_2 &= \Delta + \Re(n_1, n_2), \\ \Re_3 &= \Re(n_1, n_2 + 1). \end{aligned} \tag{18}$$

Having obtained the density matrix $\hat{\rho}(t)$ we are therefore in a position to discuss the properties of the atom and the field. Furthermore, from the form of the density matrix $\hat{\rho}(t)$ we can gain valuable insights into the problem. For example, from the solution (12) we find the atomic occupation probabilities are given by

$$\begin{aligned} P_a &= \sum_{n_1,n_2=0}^{\infty} |A(n_1, n_2; t)|^2, \\ P_b &= \sum_{n_1,n_2=0}^{\infty} |B(n_1, n_2; t)|^2, \\ P_c &= \sum_{n_1,n_2=0}^{\infty} |C(n_1, n_2; t)|^2. \end{aligned} \tag{19}$$

By using equation (5) and comparing it with equation (9), we obtain $n_1 = n+q, n_2 = n$ and we can express b_{n_1,n_2} as,

$$b_{n_1,n_2} = b_{n+q,n} = N_q \frac{|\zeta|^n}{\sqrt{n!\,(n+q)!}}\delta_{n_1,n+q}\delta_{n_2,n}. \tag{20}$$

Employing the reduced field density operator given by equation (12), we shall investigate the properties of the entropy in the next section.

The experimental feasibility of this model involving a two-mode high-Q cavity has been more or less tacitly assumed by many authors [27]. It is worthwhile remarking that investigating such models goes beyond an intrinsic theoretical interest because a new generation of high-Q electromagnetic cavities, covering a wide wavelength range, are today realizable [28]. In particular, the continuous development of new and improved materials expected to lead to fabrication of three-dimensional photonic band gap systems passing few isolated high-Q resonant field modes [29]. Such technological advances make the exploration of novel quantum electrodynamics phenomena in condensed matter systems quite attractive [30]. Intensity-dependent models like our model might be of interest in this context [31].

5 Degree of entanglement

The physical essence of entanglement consists in the existence of quantum correlations between the individual parts of a composite system that have interacted once in the past but are no longer interacting. Formally, these correlations are caused by the combination of the superposition principle in quantum mechanics with the tensor product structure of the space of states [32]. In this paper, we use the quantum field entropy as a measurement of the degree of entanglement between the field and the atom of the system under consideration. Quantum mechanically, the entropy is defined as

$$S = -\mathrm{Tr}\{\rho \ln \rho\}, \tag{21}$$

where ρ is the density operator for a given quantum system and we have set Boltzmann's constant $k = 1$. If ρ describes a pure state, then $S = 0$, and if ρ describes a mixed state, then $S \neq 0$. Consider F and A that interact with each other. How are the entropies of these systems related to the entropy of the composite system that comprises them both? The answer to this question was listed by the Araki-Lieb theorem [33]. Let S_F and S_A denote the entropies of the two interacting systems and let S denotes the entropy of the composite system. Araki and Lieb showed that these entropies satisfy the "triangle inequalities" $|S_A - S_F| \leq S \leq S_A + S_F$. Quantum entropies are generally difficult to compute because they involve the diagonalization of large (and, in many cases, infinite dimensional) density matrices. Thus explicit illustrations of the inequalities $|S_A - S_F| \leq S \leq S_A + S_F$ are difficult to come by. Knight and co-workers [34] gave a nice illustration of these inequalities in the context of the Jaynes-Cummings model. The entropies of the atom and the field, when treated as a separate system, are defined through the corresponding reduced density operators by

$$S_{A(F)} = -\mathrm{Tr}_{A(F)}\{\rho_{A(F)} \ln \rho_{A(F)}\}\cdot \qquad (22)$$

The quantum dynamics described by the Hamiltonian (8) leads to an entanglement between the field and the atom. In this paper, we use the field entropy as a measurement of the degree of entanglement between the field and the atom of the system under consideration. In order to derive a calculation formalism of the field entropy, we must obtain the eigenvalues of the reduced field density operator. By using equation (12) the reduced field density operator $\hat{\rho}_f(t) = \mathrm{Tr}_{\mathrm{atom}}\hat{\rho}(t)$ in the following form

$$\hat{\rho}_f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big[\Im_1(n)\Im_1^*(m)|\Theta_1(n)\rangle\langle\Theta_1(m)| + \Im_2(n)\Im_2^*(m)|\Theta_2(n)\rangle\langle\Theta_2(m)| + \Im_3(n)\Im_3^*(m) |\Theta_3(n)\rangle\langle\Theta_3(m)|\Big], \qquad (23)$$

where

$$\begin{pmatrix} |\Theta_1(n)\rangle \\ |\Theta_2(n)\rangle \\ |\Theta_3(n)\rangle \end{pmatrix} = \begin{pmatrix} |n_1+1, n_2\rangle \\ |n_1, n_2\rangle \\ |n_1, n_2+1\rangle \end{pmatrix}, \qquad (24)$$

one can write equation (23) in the following form

$$\hat{\rho}_f(t) = |C(t)\rangle\langle C(t)| + |S(t)\rangle\langle S(t)| + |R(t)\rangle\langle R(t)|, \qquad (25)$$

where the bimodal field states $|C(t)\rangle, |S(t)\rangle$ and $|R(t)\rangle$ are given by

$$|C(t)\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \Im_1(n)|\Theta_1(n)\rangle,$$

$$|S(t)\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \Im_2(n)|\Theta_2(n)\rangle,$$

$$|R(t)\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \Im_3(n)|\Theta_3(n)\rangle\cdot \qquad (26)$$

To calculate the various field eigenstates in a simple way, we assume that the state equation can be written in the following form

$$|\psi_f(t)\rangle = \gamma_1|C(t)\rangle + \gamma_2|S(t)\rangle + \gamma_3|R(t)\rangle. \qquad (27)$$

If we apply the density matrix given by equation (25) to the state equation (27), we find that

$$\begin{aligned}\hat{\rho}_f(t)|\psi_f(t)\rangle = &\left(\langle C(t)|C(t)\rangle + \langle C(t)|S(t)\rangle\frac{\gamma_2}{\gamma_1} + \langle C(t)|R(t)\rangle\frac{\gamma_3}{\gamma_1}\right)\gamma_1|C(t)\rangle \\ &+\left(\langle S(t)|C(t)\rangle\frac{\gamma_1}{\gamma_2} + \langle S(t)|S(t)\rangle + \langle S(t)|R(t)\rangle\frac{\gamma_3}{\gamma_2}\right)\gamma_2|S(t)\rangle \\ &+\left(\langle R(t)|C(t)\rangle\frac{\gamma_1}{\gamma_3} + \langle R(t)|S(t)\rangle\frac{\gamma_2}{\gamma_3} + \langle R(t)|R(t)\rangle\right)\gamma_3|R(t)\rangle\cdot\end{aligned} \qquad (28)$$

Consequently for $|\psi_f(t)\rangle$ to be an eigenstate of $\hat{\rho}_f(t)$ for the eigenvalue $\lambda_f(t)$, we must have the relation

$$\begin{aligned}\lambda_f(t) &= \langle C(t)|C(t)\rangle + \langle C(t)|S(t)\rangle\frac{\gamma_2}{\gamma_1} + \langle C(t)|R(t)\rangle\frac{\gamma_3}{\gamma_1} \\ &= \langle S(t)|S(t)\rangle + \langle S(t)|C(t)\rangle\frac{\gamma_1}{\gamma_2} + \langle S(t)|R(t)\rangle\frac{\gamma_3}{\gamma_2} \\ &= \langle R(t)|R(t)\rangle + \langle R(t)|C(t)\rangle\frac{\gamma_1}{\gamma_3} + \langle R(t)|S(t)\rangle\frac{\gamma_2}{\gamma_3}\cdot\end{aligned} \qquad (29)$$

Then the eigenvalues of the density matrix of the field are given by

$$\begin{aligned}\lambda_f^{(1)}(t) &= \frac{\vartheta_1}{3} - \frac{2}{3}\left(\sqrt{\vartheta_1^2 - 3\vartheta_2}\right)\cos\varepsilon, \\ \lambda_f^{(2)}(t) &= \frac{\vartheta_1}{3} + \frac{1}{3}(\cos\varepsilon + \sqrt{3}\sin\varepsilon)\left(\sqrt{\vartheta_1^2 - 3\vartheta_2}\right), \\ \lambda_f^{(3)}(t) &= \frac{\vartheta_1}{3} + \frac{1}{3}(\cos\varepsilon - \sqrt{3}\sin\varepsilon)\left(\sqrt{\vartheta_1^2 - 3\vartheta_2}\right),\end{aligned} \qquad (30)$$

where

$$\begin{aligned}\varepsilon &= \frac{1}{3}\arccos\left(\frac{Q}{W}\right), \\ Q &= 9\vartheta_1\vartheta_2 - 2\vartheta_1^3 - 27\vartheta_3, \\ W &= 2\sqrt{(\vartheta_1^2 - 3\vartheta_2)^3},\end{aligned} \qquad (31)$$

with

$$\begin{aligned}\vartheta_1 &= -(\langle C(t)|C(t)\rangle + \langle S(t)|S(t)\rangle + \langle R(t)|R(t)\rangle), \\ \vartheta_2 &= \langle C(t)|C(t)\rangle\langle R(t)|R(t)\rangle + \langle C(t)|C(t)\rangle\langle S(t)|S(t)\rangle \\ &\quad + \langle S(t)|S(t)\rangle\langle R(t)|R(t)\rangle - |\langle S(t)|R(t)\rangle|^2 \\ &\quad - |\langle C(t)|S(t)\rangle|^2 - |\langle C(t)|R(t)\rangle|^2, \\ \vartheta_3 &= \langle C(t)|C(t)\rangle\langle R(t)|R(t)\rangle\langle S(t)|S(t)\rangle + \langle C(t)|S(t)\rangle \\ &\quad \times\langle R(t)|C(t)\rangle\langle S(t)|R(t)\rangle - \langle C(t)|C(t)\rangle|\langle S(t)|R(t)\rangle|^2 \\ &\quad - \langle R(t)|R(t)\rangle|\langle C(t)|S(t)\rangle|^2 \\ &\quad - \langle S(t)|S(t)\rangle|\langle R(t)|C(t)\rangle|^2.\end{aligned} \qquad (32)$$

The field entropy $S_f(t)$ may be expressed in terms of the eigenvalues $\lambda_f^{(i)}(t)$ for the reduced field density operator as,

$$S_f(t) = -\ln\left(\left[\lambda_f^{(1)}(t)\right]^{\lambda_f^{(1)}(t)} \times \left[\lambda_f^{(2)}(t)\right]^{\lambda_f^{(2)}(t)} \times \left[\lambda_f^{(3)}(t)\right]^{\lambda_f^{(3)}(t)}\right). \quad (33)$$

In the case of a disentangled pure joint state $S_f(t)$ is zero, and for maximally entangled states it gives $\ln 3$. Since we cannot obtain a simple analytical expression for the series included in equation (33) (because of the higher dimensionality of the problem), the numerical approach becomes indispensable. In the following section, we shall present our results obtained from a numerical solution of the evolution equations (33) with emphasis on the effect of the nonlinearities on the behavior of the system under consideration.

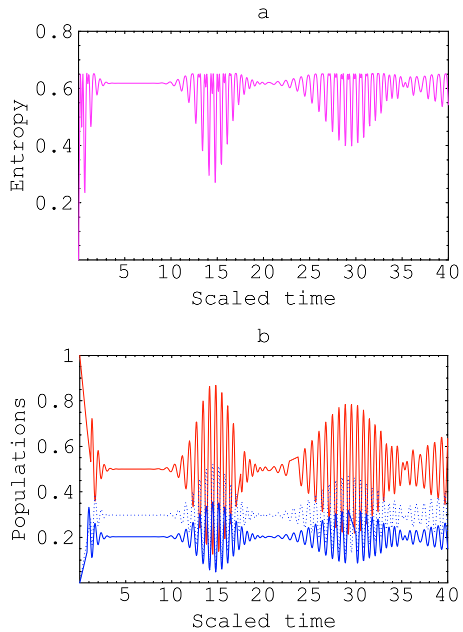

Fig. 1. (a) The evolution of the field entropy S_F as a function of the scaled time λt with $q = 4$, $\zeta = 10$, $\Re(n_1, n_2) = 0$, $f_i(\hat{n}_i) = 1$ and $\Delta/\lambda = 0$. (b) Atomic level occupation probabilities, (sold line) the upper state $P_1(t)$, the first lower state (doted line) $P_0(t)$ and the second lower state (doted-dashed line) $P_2(t)$ as functions of the scaled time λt for the system under consideration.

6 Results and discussions

Given the impressive technological advances in several experimental areas of quantum optics, condensed matter, atomic physics, etc., it is nowadays possible to realize a system of two interacting degrees of freedom and watch the time evolution of the corresponding entanglement process [36]. It is therefore also of importance to understand the entanglement process in simple Hamiltonian systems. Hamiltonian systems with two degrees of freedom often present a very rich dynamics, which in many cases is not yet completely understood from a general point of view. In particular, if the interaction is nonlinear the system may present chaotic behavior in the classical limit. The consequences of this fact to the quantum dynamics is yet an unsettled issue. A step in this direction was taken a few years ago, as it was conjectured that "the rate of entropy production can be used as an intrinsically quantum test of the chaotic *versus* regular nature of the evolution" [37]. The idea has been tested in some models [38]. In this section we shall discuss and analyze the behavior of the field entropy for the present model. Our intention here is to show how the quantum field entropy and the populations are influenced by different kinds of nonlinearities, so we next present results that clearly show this influence. Initially we fix the number of the correlated two-mode coherent state at $q = 4$ and $\zeta = 10$, to analyze the effects resulting from variation in the nonlinearities of both the bimodal field and the intensity coupling.

In Figure 1a we display the time evolution of the field entropy as a function of the scaled time λt, of a three-level atom interacting with two-mode pair coherent states in the absence of both nonlinearities and detuning *i.e.* ($\Re(n_1, n_2) = 0, f_i(n_i) = 1, \Delta = 0$). It is observed that the maximum and minimum values of the field entropy are achieved during the initial stage of the time evolution. At later time the field and the atom are strongly entangled. The probability amplitudes given in expressions (19) are evaluated numerically and the results given in Figure 1b. On the interval $0 \leq \lambda t \leq 5$ the Rabi oscillation in the atomic level occupation probabilities almost collapse, while on the interval $10 \leq \lambda t \leq 30$ one encounters two well-separated revivals. The amplitude of the Rabi oscillations attains a maximum at the so-called revival times [35]. For longer times the entropy as well as the atomic level occupation probabilities fluctuate in an irregular manner. It is worth noting that the state into which the cavity collapses due to a successful measurement of the atomic state exhibits maximum entanglement regardless both of the initial equal populations given to the two cavity modes and the duration of interaction.

Figures 2 and 3 are representing different values of the intensity coupling, where the values of the parameters $f_i(n_i)$ are equal to $\sqrt{n_i}$, for Figures 2 and $1/\sqrt{n_i}$ for Figure 3, and the other parameters have the same values as in Figure 1. One observes that the entropy shows rapid oscillations (see Fig. 2) but the situation is completely changed when we consider $f(n_i) = 1/\sqrt{n_i}$, the entropy has zero values in a regular manner at period $\sim \pi/\sqrt{2}$. The effect of the appearance of n has been compensated for. This is particularly because of the nonlinear nature of the coupling in this model which results in the Rabi frequency being proportional to the photon number. In the absence of the nonlinearities, and for zero detuning the Rabi oscillation is given by $\Omega_n = \sqrt{2(n+1)}$. The spectrum of the Rabi frequencies is nonlinear in n. Let us treat this frequency as a continuous quantity and expand

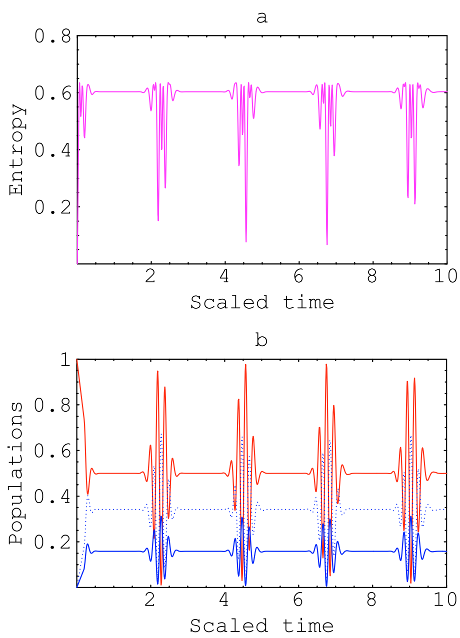

Fig. 2. (a) The evolution of the field entropy S_F as a function of the scaled time λt with $q = 4$, $\zeta = 10$, $\Re(n_1, n_2) = 0$, $f_i(\hat{n}_i) = \sqrt{n_i}$ and $\Delta/\lambda = 0$. (b) Atomic level occupation probabilities, (sold line) the upper state $P_1(t)$, the first lower state (doted line) $P_0(t)$ and the second lower state (doted-dashed line) $P_2(t)$ as functions of the scaled time λt for the system under consideration.

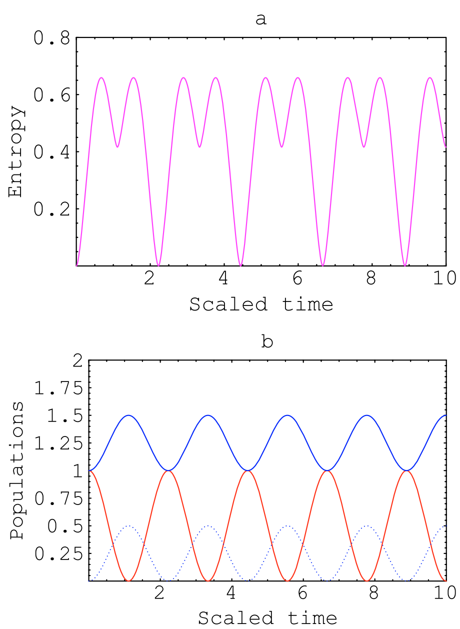

Fig. 3. (a) The evolution of the field entropy S_F as a function of the scaled time λt with $q = 4$, $\zeta = 10$, $\Re(n_1, n_2) = 0$, $f_i(\hat{n}_i) = 1/\sqrt{n_i}$, and $\Delta/\lambda = 0$. (b) Atomic level occupation probabilities, (sold line) the upper state $P_1(t)$, the first lower state (doted line) $P_0(t)$ and the second lower state (doted-dashed line) $(P_2(t) + 1)$ as functions of the scaled time λt for the system under consideration.

the dispersion curve Ω_n around the point $\bar{n} = |\zeta|^2$. Let us write

$$\Omega_n = \Omega_{\bar{n}} + \Omega_{\bar{n}}^{(1)}(n - \bar{n}) + \Omega_{\bar{n}}^{(2)}(n - \bar{n})^2 + ..., \qquad (34)$$

where $\Omega_{\bar{n}}^{(r)} = \frac{1}{k!} \left. \frac{\mathrm{d}^k \Omega_n}{\mathrm{d}n^k} \right|_{n=\bar{n}}$. The first term of the Ω_n expansion is responsible for rapid oscillations of the model while the remaining terms are responsible for their envelope. In general, if only the first-order derivative of such an expansion were different from zero, the collapses and revivals of the oscillations would be perfectly periodic (linear or harmonic approximation) which is the case for both $f(n_i) = \sqrt{n_i}$ or $1/\sqrt{n_i}$ as well. If higher-order terms in Ω_n are nonzero, but the nonlinearity of the frequency spectrum is slight, they spread the revivals arising from the linear expansion and, in particular, lead to their incompleteness, overlapping and a ringing structure. In turn, if the influence of the higher-order terms in Ω_n is significant, it may totally wash out collapses and revivals of the model. Comparing the behavior in Figure 2, where we set the intensity coupling constants $f(n_i) = \sqrt{n_i}$, with cases considered in Figure 1, we may say that the effect of the intensity coupling is rather different, where the oscillating period for $f(n_i) = 1$ is longer than that of $f(n_i) = \sqrt{n_i}$ case. Also, one can realize that, periodical changes always occurring in the entropy as a common property in this case. This should be expected as resultant of the existence of the periodic functions in the expression of the entropy.

Now we will examine the precise role that the nonlinear medium (Kerr-type) actually plays in the entanglement degree. When the nonlinearity takes place and starts to affect the system, we can easily realize a lot of changes occurring in the entropy as well as in the atomic occupation probabilities. For example we set $R(\hat{n}_1, \hat{n}_2) = \chi_1 n_1(n_1 - 1) + \chi_2 n_2(n_2 - 1)$, where $\chi_i, (i = 1, 2)$ (which are related to the third-order nonlinear susceptibilities for the processes of self-phase-modulation of the two modes). In fact the optical Kerr effect is one of the most extensively studied phenomenon in the field of nonlinear optics because of its various applications [39–43]. In Figure 4, we set $f(\hat{n}_i) = 1$, then by taking $\chi_1 = \chi_2 = \chi$ and $\chi/\lambda = 0.5$, while the difference number of two-mode photon $q = 4$ and $\zeta = 10$, we show that a nonlinear interaction of the Kerr-like medium with the field mode leads to increasing values of the minimum entropy and of the sustainment time of the maximum entropy. In this case, the field and the atom almost retain a strong entanglement in the time evolution process. With the increase of the nonlinear interaction of the Kerr-like medium with field mode, the maximum value of the entropy becomes less and less (*i.e.* the increasing of the nonlinear interaction of the Kerr-like medium with field mode leads to a decrease of the maximum value of the entanglement degree). This means that with increasing the constant χ there exists an enhancement of the energy exchange between atom and optical field. We find in particular that the Kerr-like medium yields the superstructure of atomic Rabi oscillation and the interaction intensity of atom-field is non-monotonically dependent on the dispersive part of the third-order nonlinearity of the Kerr-like medium.

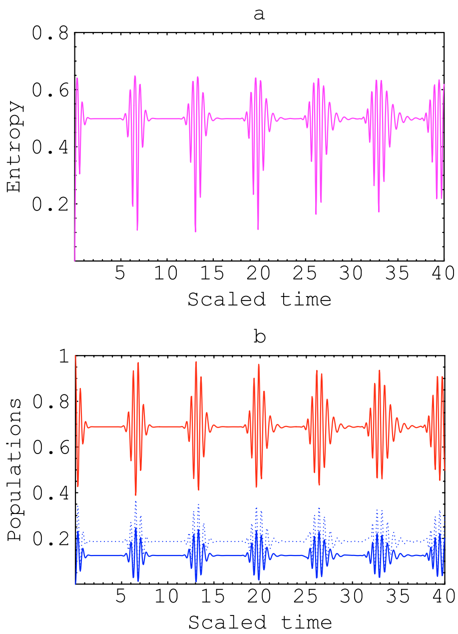

Fig. 4. (a) The evolution of the field entropy S_F as a function of the scaled time λt with $q = 4$, $\zeta = 10$, $\Re(n_1, n_2) = \chi_1 n_1(n_1 - 1) + \chi_2 n_2(n_2 - 1)$, $(\chi_1 = \chi_2 = \chi = 0.5)$, $f_i(\hat{n}_i) = 1$, and $\Delta/\lambda = 0$. (b) Atomic level occupation probabilities, (sold line) the upper state $P_1(t)$, the first lower state (doted line) $P_0(t)$ and the second lower state (doted-dashed line) $P_2(t)$ as functions of the scaled time λt for the system under consideration.

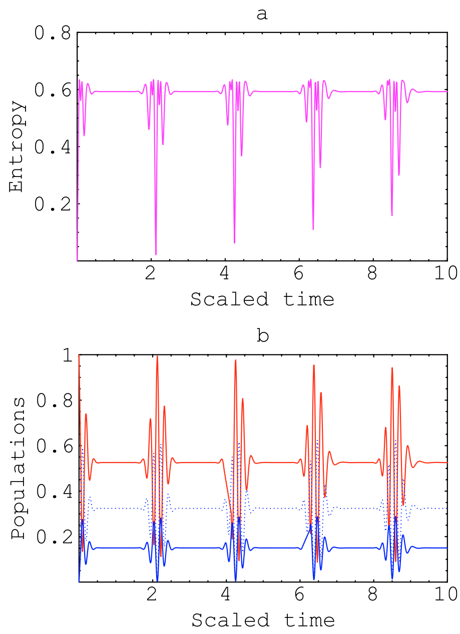

Fig. 5. (a) The evolution of the field entropy S_F as a function of the scaled time λt with $q = 4$, $\zeta = 10$, $\Re(n_1, n_2) = \chi_1 n_1(n_1-1)+\chi_2 n_2(n_2-1)$, $(\chi_1 = \chi_2 = \chi = 0.5)$, $f_i(\hat{n}_i) = \sqrt{n_i}$ and $\Delta/\lambda = 0$. (b) Atomic level occupation probabilities, (sold line) the upper state $P_1(t)$, the first lower state (doted line) $P_0(t)$ and the second lower state (doted-dashed line) $P_2(t)$ as functions of the scaled time λt for the system under consideration.

Comparing the behavior in Figure 5 where $f(n_i) = \sqrt{n_i}$ and Figure 6 where $f(n_i) = 1/\sqrt{n_i}$ in the presence of the nonlinear medium $\chi/\lambda = 0.5$, which implies that the effects on the entropy of both specific intensity-dependent nonlinear and the nonlinear medium can be counterbalanced in some special case. The result in Figure 6 is in marked contrast to the situation of $f(n_i) = 1/\sqrt{n_i}$ in absence of the nonlinear medium, where the maximum value of the entropy is decreased dramatically, also we have shown here the periodic oscillations occur in all cases. This difference reflects the various influences of intensity-dependent media on the interaction between atom and field. The evolution of the three-level atom is governed by the collapse and revivals of the both slow and fast oscillations. The atom and the field never stop exchanging energy, so convergence to a final state does not take place. In Figure 6, we note that the amplitude of the field entropy decreases. It is evident that the field and the atom are in pure states when the Kerr-like effect increase further. This result corresponds to the fact that in the limit for the very strong nonlinear interaction of the Kerr-like medium with the field mode, the field and the atom are almost decoupled, which preserves the field entropy's tending to zero. Also, from our further calculations, which are not displayed here, we see how the field entropy and the atomic occupation probabilities depend on the detuning parameters at a given time. By augmenting detuning parameter we bring about qualitative changes in the evolution.

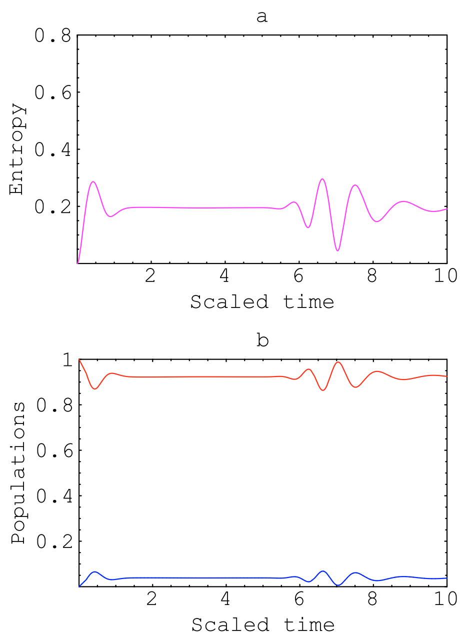

Fig. 6. (a) The evolution of the field entropy S_F as a function of the scaled time λt with $q = 4$, $\zeta = 10$, $\Re(n_1, n_2) = \chi_1 n_1(n_1 - 1) + \chi_2 n_2(n_2 - 1)$, $(\chi_1 = \chi_2 = \chi = 0.5)$, $f_i(\hat{n}_i) = 1/\sqrt{n_i}$ and $\Delta/\lambda = 0$. (b) Atomic level occupation probabilities, (sold line) the upper state $P_1(t)$, the first lower state (doted line) $P_0(t)$ and the second lower state (doted-dashed line) $P_2(t)$ as functions of the scaled time λt for the system under consideration.

It is observed that the first maximum value of the field entropy decreases, but the period of revivals becomes longer and the time area of vibration of the entropy is compressed. On the other hand, the effect of the detuning on the oscillation frequency and on the time of the revivals after the first one do not seem regular. This means that the dependence on the detuning can be rather complicated, and further study is necessary before more definite conclusions can be made. One may eliminate adiabatically the upper atomic level, and reduce the three-level system to an effective two-level one. From this point of view, the transition of the electron can be considered as existing only between the states $|b\rangle$ and $|c\rangle$. The above analysis means that by increasing the detuning the Kerr-like medium effect results in a peculiar phenomenon that the transition of an electron between the $|b\rangle$ and $|c\rangle$ atomic states. This new effect may be interpreted as due to the dynamical Stark shift induced by nonlinear interaction of Kerr-like medium [39]. To see this, we rewrite the generalized Rabi frequency of the three level atom as $\Omega_{ij}(\chi/\Delta) = \beta_i(\chi/\Delta) - \beta_j(\chi/\Delta)$, $i, j (i < j) = 1, 2, 3$. For $\chi \ll \Delta$, one expand this expression as a parameter as a power series of χ/Δ as $\Omega_{ij}(\chi/\Delta) = \Omega_{ij}(0) + (\chi/\Delta)\Omega^{\backslash}_{ij}(0) + 0(\chi/\Delta)^2$, where $\Omega^{\backslash}_{ij}(0) = \partial\Omega_{ij}(\chi/\Delta)/\partial(\chi/\Delta)$. The term $(\chi/\Delta)\Omega^{\backslash}_{ij}(0)$ is seen to be the dynamical Stark shift, however, induced by Kerr-like medium effect.

It may be worth pointing out that the quantum entanglement in the present model arises from interaction between the two subsystems and therefore depends on the interaction strength between the two, in contrast to quantum entanglement discussed in the context of quantum information theory which is traditionally based on Bell-type bipartite spin-singlet states (for a recent review see [44]). In this case, for the pure state density matrix associated with a single Bell state, the conditional entropy is trivially seen to be negative and hence the Bell state is super correlated. A different type of entropic analysis based on Jaynes maximum entropy principle for this system has been given recently [45].

The generation of maximally entangled field state between two cavities such that if one cavity has one photon then the other will be in vacuum, the atom after its interaction with the cavity fields, is required to be detected in ground state $|c\rangle$ or $|b\rangle$. This leads to the condition that probability amplitudes of the states $|0, 1; b\rangle$, and $|1, 0; c\rangle$ are equal. The total probability of detecting the atom in $|b\rangle$ and $|c\rangle$ states is determined as $P_g = \frac{1}{2}(\sin^2 \lambda_1\mu_n t + \sin^2 \lambda_2\mu_n t)$ [46]. This probability becomes maximum when the time of interaction of atom with mode A and mode B is $m\pi/2\lambda_1\mu_n$ and $n\pi/2\lambda_2\mu_n$, respectively (in this case $\mu_n = 1$). Here, m and n are odd integer numbers. Hence, in order to generate two mode entanglement the time of interaction of the atom with the cavity is odd integer multiple of half of the Rabi cycle. This ensures that the cavity will obtain one photon in either of the two modes when atom is detected in ground state after its propagation through the cavity. As a result the atom leaves the cavity in upper state and develops an entangled state between the two cavity modes. The interaction times of the atom with the two modes of the cavity field would be different because of the different coupling constants of each mode of radiation field. These interaction times of atom in the cavity can be controlled by using a velocity selector before the cavity and then applying Stark field adjustment so that atom becomes resonant with the cavity field modes only for the suggested amount of time in each mode of the cavity field.

We may prepare the initial state as the one in equation (9), but with $n_1 = 1$ (the first cavity field has one photon) and $n_2 = 0$ (the second cavity field in the vacuum state). After having detected the atom in the internal state, the resulting state will be

$$|\psi(0)\rangle = \gamma_1|1\rangle \otimes |0\rangle \otimes |b\rangle + \gamma_2|0\rangle \otimes |0\rangle \otimes |a\rangle + \gamma_3|0\rangle \otimes |1\rangle \otimes |c\rangle \cdot \quad (35)$$

We can detect the atom in the $|c\rangle$ state and choose $\gamma_2 = 0$ and adjust the time to get $(|0\rangle \otimes |0\rangle \pm |1\rangle \otimes |1\rangle)$, or take $\gamma_1 = 0$ and adjust it to get $(|0\rangle \otimes |1\rangle \pm |0\rangle \otimes |0\rangle)$, which is also a Bell-type state involving the quantized cavity fields. More general states could also be generated, depending on the initial conditions. Note that entanglement here involves states belonging to different kinds of physical systems (although both subspaces have infinite dimension), but because of their nature new possibilities for quantum information processing might arise. In order to realize the suggested scheme in laboratory experiment within microwave region, we may consider slow Rb atoms in higher Rydberg states which have life time of the order of few milliseconds [47]. These slow atoms, initially pumped to high Rydberg states, pass through a high-Q superconducting cavity of dimension of a few centimeters with a velocity of around 400 m/s [47–49]. The interaction times of atom with cavities come out to be of the order of few tens of microseconds which is far less than the cavity life time. The high-Q cavities of life time of the order of millisecond are being used in recent experiments [49]. The interaction time of the atom with different cavities can be controlled by using a velocity selector and applying Stark field adjustment in different cavities in order to make the atom resonant with the field for right amount of time. As this entanglement remains only for the cavity life time period so any application regarding this entangled state should be accomplished during this period.

It is important moreover to underline that the success of the procedure reported in this paper for generating maximally entangled bimodal cavity field having the form (38), is strictly related to the capacity of preparing the atom-field system in the state expressed by equation (12). As far as the atomic initial state, it is well known that the Ramsey zone method, currently used in laboratory for mixing two atomic states, is very efficient. On the other hand, it is possible to prepare the cavity field in an equally intensity bimodal Fock state, following, for example, an experimental scheme very recently presented in literature [50], it is shown that, taking into account important technological limits of the apparatus currently used in laboratory, the probability of realizing the state $|n, n\rangle$, decreases with n maintaining however values of experimental interest in correspondence to $n \sim 10$. It is important to observe, at this point, that, notwithstanding the high values of the quality factor Q of the resonators today available, it could be illegitimate to neglect the cavity losses when n is too large. These considerations suggest that from an experimental point of view it should be better to choose initial conditions such that the number of photons contained in both cavity modes at $t = 0$ does not exceed 10.

Here we may refer to an interesting work given in reference [51] where the authors considered a three-level atom interacting with a single mode and presented a scheme for preparing entangled coherent states based on the atom-cavity mode Raman interaction. Furthermore, they discussed a method for generating multi-mode entangled coherent states.

7 Conclusion

To summarize, we have discussed the interaction of a three-level atom bimodal field system, taking into account arbitrary forms of nonlinearities of both the field and the intensity-dependent atom-field coupling. The work here extends previous studies in this context [16,17,46,51,52]. Analytical expression for the density operator of the system is derived. Whilst the model was quite general, we have chosen intentionally to study specific kinds of nonlinearities of both the field and the intensity-dependent atom-field coupling, for which the relevant experiments are available, so as to make quantitative predictions. In particular, we have explored the influence of the various parameters of the system on the field entropy, entanglement and the evolution of atomic occupation probabilities. We have demonstrated that the entanglement degree is extremely sensitive to different kinds of nonlinearities of both the field and the intensity-dependent atom-field coupling. We have shown numerically that the time evolution of the quantum field entropy and atomic occupation probabilities depend sensitively on the nonlinearities. We find in particular that the nonlinear medium yields the superstructure of atomic Rabi oscillation and the interaction intensity of atom-field is non-monotonically dependent on the coupling constant.

An idealized situation when the cavity losses are negligible is considered here. However, for the case of real experiment the losses must be introduced. It can be expected that for a non-ideal but high-quality cavity our results are of relevance in the case the Hamiltonian is appropriate for the experimental setup. We considered atomic occupation probabilities and found that the phenomenon of periodic collapse and revival occurs; it is however a short-lived phenomenon due to the effects of nonlinearity. It is found that entanglement is affected strongly when a nonlinear medium is taken into account in the presence of the intensity coupling such that $f(n_i) = 1/\sqrt{n_i}$. When the nonlinear interaction of the Kerr-type medium is very strong, it leads to a decrease of the field entropy. For the intensity-dependent atom-field coupling $f(n_i) = \sqrt{n_i}$ the entropy as well as the atomic occupation probabilities exhibit strong regular oscillation.

We have shown that it is possible to generate Bell-type states having rather simple initial state preparation. To make the problem more realistic the atom decay and cavity decay should be taken into account. We hope to report on such issues in a forthcoming paper. Also, it is interesting to point out that these results are quite general and can be applied to other systems which involve more atomic levels or more field modes to exhibit large entanglement under certain conditions.

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